

# AUTOMATICALLY REDUCED DEGENERATIONS OF AUTOMATICALLY NORMAL VARIETIES

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ABSTRACT. Let  $F$  be a flat family of projective schemes, whose geometric generic fiber is reduced and irreducible. We give conditions on a special fiber (a “limit” of the family) to guarantee that it too is reduced. These conditions often imply also that the generic fiber is normal. The conditions are particularly easy to check in the setup of a “geometric vertex decomposition” [Knutson-Miller-Yong ’07].

The primary tool used is the corresponding limit *branchvariety* [Alexeev-Knutson ’06], which is reduced by construction, and maps to the limit subscheme; our technique is to use normality to show that the branchvariety map must be an isomorphism.

As a demonstration, we give an essentially naïve proof that Schubert varieties in finite type are normal and Cohen-Macaulay. The proof does not involve any resolution of singularities or cohomology-vanishing techniques (e.g. appeal to characteristic  $p$ ).

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## 1. STATEMENT OF RESULTS

Let  $F \subseteq \mathbb{P}^n \times S$  be a closed subscheme, flat over  $S$ , considered as a family over  $S$  of projective schemes. If  $S$  is irreducible, we can speak of the generic fiber of  $F$ , which throughout this paper we assume to be (geometrically) reduced.

**1.1. General limits.** It is frequently useful to be able to guarantee that a particular fiber  $F_o$  over a point  $o \in S$  is reduced. Often its underlying set may be easy to calculate, but we may only be able to check its reducedness generically, or in small codimension.

**Reduced Limit Lemma.** *Let  $F \subseteq \mathbb{P}^n \times S$  be a flat family of  $d$ -dimensional projective schemes (over a fixed Noetherian base scheme). Let  $S$  be irreducible and normal, and assume the generic fiber of  $F \rightarrow S$  is irreducible (or at least, equidimensional and connected in codimension 1) and geometrically reduced.*

*Let  $F_o$  denote the fiber over a point  $o \in S$ . Let  $A_1, A_2, \dots, A_k$  be the components of its reduction, automatically of dimension  $d$ . Assume that  $F_o$  is generically geometrically reduced and each  $A_i$  is normal, and that at least one of the following holds:*

- (1)  *$F_o$  is irreducible ( $k = 1$ ),*
- (2)  *$F_o$  has only two geometric components ( $k = 2$ ), and  $A_1 \cap A_2$  is reduced and irreducible,*
- (3)  *$F_o$  is reduced through codimension 1, and for each  $i$ ,  $A_i \cap (A_1 \cup \dots \cup A_{i-1})$  is equidimensional of dimension  $d - 1$ , and reduced. (This may involve reordering the  $\{A_i\}$ .)*

*Then  $F_o$  is reduced.*

*In case (1), the generic fiber is irreducible and normal. In case (2), if it is irreducible, it is normal. In cases (2) and (3) it at least satisfies Serre's condition  $S_2$ .*

Case (1) is implied by a much older local version from [H58]; see also [Ko95] and the references therein. The conditions in the lemma seem difficult to weaken in cases (2) and (3), as a few near-counterexamples may help demonstrate, though a local version perhaps may be achievable using the results of [BLR95]. In each of the following examples all conditions other than the italicized one hold, but  $F_o$  is not reduced.

- *Generic fiber reducible.* Let  $F_t$  be the union of two skew lines in  $\mathbb{P}^3$  at distance  $t$  from one another; at  $t = 0$  let them cross at one point. In  $F_o$ , there is an embedded point at the crossing.
- *Special fiber not generically reduced.* Let a smooth plane conic degenerate to a double line. Then all other conditions of case (1) hold.
- *$A_1 \cap A_2$  reducible.* Let  $X$  be a twisted cubic curve in  $\mathbb{P}^3$ , degenerating to a planar union of a line and a conic, with an embedded point at one of the two points of intersection. Then all other conditions of case (2) hold.
- *Nonprojective fibers.* From the previous example, excise a generic  $\mathbb{P}^2$  passing through the other (the reduced) point of intersection. Then  $X$  is a twisted cubic in  $\mathbb{A}^3$ , degenerating to the union of a line and conic in  $\mathbb{A}^2$ , with an embedded point at the single point of intersection.

The third case in the Reduced Limit Lemma contains the first, as  $k = 1$ . At  $k = 2$  it is slightly different from the second case; it is more generally applicable (in not requiring  $A_1 \cap A_2$  irreducible) but harder to apply, in that one is required to check reducedness in codimension 1 by other means.

The condition “ $A_i \cap (A_1 \cup \dots \cup A_{i-1})$  is equidimensional of dimension  $d-1$ ” also makes sense when the  $\{A_i\}$  are the facets of a simplicial complex; in that theory an ordering with this property is called a *shelling*.

We will recall what we need about Serre’s conditions  $S_k$  in section 2.1. The conclusion of the Reduced Limit Lemma that the generic fiber is  $S_2$  has a simple extension:

**Lemma 1.** *Assume the setup of the Reduced Limit Lemma (so in particular,  $F_0$  is reduced). Ask in addition that each component  $A_i$  of the special fiber  $F_0$  is  $S_k$ , and that each  $A_i \cap (A_1 \cup \dots \cup A_{i-1})$  is  $S_{k-1}$ . Then  $F_0$  and the generic fiber are  $S_k$ .*

*In particular, if each  $A_i$  and  $A_i \cap (A_1 \cup \dots \cup A_{i-1})$  are Cohen-Macaulay, then  $F_0$  and the generic fiber are Cohen-Macaulay.*

In the applications envisioned by the author (one of which will occupy section 4), one starts with a general fiber, constructs a one-parameter family over a punctured disc  $S \setminus o$ , and fills in the limit  $F_0$  by taking a certain closure. (Note that this construction requires that the family be embedded, in order to have somewhere to take a closure.) The Reduced Limit Lemma is then invoked to study this *automatically flat* limit. A slightly different point of view is taken in [Ko95], where one is given the family (so no embedding is necessary) and one wants criteria to check whether it is flat.

**1.2. Geometric vertex decompositions.** We now describe a very specific sort of family which we proved some results about already in [KMY07]. In this restricted case the same techniques yield a stronger result, as the conditions to check are particularly simple.

Let  $H \oplus L$  be a vector space, where  $L$  is one-dimensional (the letters are for Hyperplane and Line). Let  $X \subseteq H \oplus L$  be a reduced, irreducible subvariety, and consider its closure  $\bar{X}$  inside  $H \times \mathbb{LP}^1$ , where  $\mathbb{LP}^1 = L \cup \{\infty\}$  denotes the projective completion of  $L$ .

Define the family

$$F := \overline{\{(h, \ell, z) : (h, z^{-1}\ell) \in \bar{X}\}} \subseteq H \times \mathbb{LP}^1 \times \mathbb{A}^1, \quad \text{considered over } \mathbb{A}^1.$$

If we let  $\mathbb{G}_m$  act on  $H \times \mathbb{LP}^1$  by scaling the second factor,  $z \cdot (\vec{h}, \ell) := (\vec{h}, z\ell)$ , then  $F_{z \neq 0} = z \cdot X$ ; every closed fiber but  $F_0$  is isomorphic to  $F_1 = \bar{X}$ . This  $F$  is automatically flat over  $\mathbb{A}^1$ .

In the case that  $F_0$  is reduced, we christened it a **geometric vertex decomposition** of  $X$  in [KMY07], as the splitting in equation (1) below is closely related to the splitting of a simplicial complex using a “vertex decomposition”.

**Geometric Vertex Decomposition Lemma.** *Let  $\bar{X} \subseteq H \times \mathbb{LP}^1$  be irreducible and geometrically reduced, with  $\Pi$  its projection to  $H$  and  $\Lambda := \bar{X} \cap (H \times \{\infty\})$ . Assume  $\Lambda \subsetneq \bar{X}$ , i.e.  $\bar{X}$  is the closure of  $X := \bar{X} \cap (H \times L)$ . Let*

$$F := \overline{\{(h, \ell, z) : (h, z^{-1}\ell) \in \bar{X}\}} \subseteq H \times \mathbb{LP}^1 \times \mathbb{A}^1.$$

*So far this is the general setup of [KMY07, theorem 2.2], which asserts that*

$$F_0 = (\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{LP}^1) \tag{1}$$

*as sets. Assume in addition that*

- (1) *the projection  $\bar{X} \rightarrow \Pi$  is generically 1:1,*
- (2)  *$\Pi$  is normal, and*
- (3)  *$\Lambda$  is geometrically reduced.*

Then equation (1) holds as schemes, i.e.  $F_0$  is reduced, a geometric vertex decomposition of  $X$ .

If  $\Pi$  and  $\Lambda$  are Cohen-Macaulay, then so are  $F_0$ ,  $\bar{X}$ , and  $X$ .

In case (2) of the Reduced Limit Lemma, we required the intersection  $A_1 \cap A_2$  to be reduced; the analogue of this in the Geometric Vertex Decomposition Lemma is the requirement that  $\Lambda$  be reduced. However, we do not need to require any analogues of the conditions that  $A_2$  be irreducible and normal ( $\Lambda$  may be neither), and all the projectivity we need is in the  $L\mathbb{P}^1$ .

As in case (2) of the Reduced Limit Lemma,  $\Lambda$  normal implies that  $\bar{X}$  is normal. Indeed, this will be the case in our application in section 4. In other situations, though,  $\Lambda$  is often only  $S_2$ , so we give a more general criterion for normality of  $\bar{X}$ :

**Lemma 2.** *Continue the situation of the Geometric Vertex Decomposition Lemma.*

*Then every component of  $\bar{X}$ 's singular locus  $\bar{X}_{\text{sing}}$  of codimension 1 in  $\bar{X}$  is of the form  $D \times L\mathbb{P}^1$ , where  $D \subseteq \Lambda_{\text{sing}}$  is codimension 1 inside  $\Lambda$ . In particular, if  $\dim \Lambda_{\text{sing}} < \dim \Lambda - 1$ , there can be no such components  $D$ .*

*If there are no such components, and  $\Lambda$  is  $S_2$ , then  $\bar{X}$  is normal. In particular,  $\Lambda$  normal implies  $\bar{X}$  also normal.*

To apply this lemma, one determines the components  $D$  of  $\Lambda_{\text{sing}}$  of codimension 1 in  $\Lambda$ , and checks that  $\bar{X}$  either does not contain or is generically nonsingular along  $D \times L\mathbb{P}^1$ .

After the proof of this lemma (in section 3) we give an example showing the criterion is necessary, in which  $\bar{X}_{\text{sing}}$  contains such a component and  $\bar{X}$  is not normal.

**1.3. Structure of the paper.** We prove all these lemmas in section 3. The crucial notion used is that of the *limit branchvariety* [AK], which is a sort of reduced avatar of the limit subscheme, but much better behaved than its simple reduction. (A similar “correction” already appears in [Ko95, remark 4.2].) In the cases at hand, though, the limit branchvariety and limit subscheme coincide, showing the limit subscheme is reduced. We recall these and other more standard notions in section 2.

In section 4 we apply these lemmas to give an inductive proof of the well-known result (see e.g. [R85]) that Schubert varieties in arbitrary finite-dimensional flag manifolds are normal and Cohen-Macaulay. In very brief, we flatly degenerate an affine patch on a Schubert variety, invoke the Geometric Vertex Decomposition Lemma to show the limit scheme is a union of two simpler patches, and use induction. In particular, the proof does not involve any resolution of singularities or cohomology-vanishing techniques (e.g. appeal to characteristic  $p$ ), and we expect it to apply to other families of subvarieties of flag manifolds.

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## 2. GEOMETRIC PRELIMINARIES

In this section we assemble some standard geometric results, with the more technical lemmas to wait until section 3.

**2.1. Serre conditions.** A scheme  $X$  is called  $S_m$  at the point  $x \in X$  if the local ring at  $x$  possesses a regular sequence of length  $m$ , perhaps after extension of the residue field. This is equivalent to the vanishing of the local cohomology groups  $H_m^i(k[X])$  for  $i < m$  [BS98, chapter 6]. If  $X$  is  $S_m$  at every point, we just say  $X$  is  $S_m$ .

These properties are related to many familiar geometric ones, particularly in tandem with the following conditions called  $\{R_j\}$ . An equidimensional scheme  $X$  is  $R_j$  if its singular locus has codimension  $> j$ . In particular,  $X$  is generically reduced iff it is  $R_0$ .

**Proposition 1.** *Let  $X$  be an equidimensional scheme.*

- (1)  $X$  is reduced iff  $X$  is  $R_0$  and  $S_1$ .
- (2)  $X$  is normal iff  $X$  is  $R_1$  and  $S_2$  (Serre's criterion).
- (3)  $X$  is Cohen-Macaulay iff  $X$  is  $S_{\dim X}$ .
- (4) If  $X$  is the union  $A \cup B$  of two closed subschemes, where  $A, B$  are  $S_k$  and  $A \cap B$  is  $S_{k-1}$ , then  $X$  is  $S_k$ .

*Proof.* (1) Exercise 11.10 of [E95].

- (2) Theorem 11.5 of [E95].
- (3) This is the usual definition.
- (4) For this we use the Mayer-Vietoris sequence on local cohomology

$$\dots \rightarrow H_m^{k-1}(k[A \cap B]) \rightarrow H_m^{k-1}(k[A] + k[B]) \rightarrow H_m^{k-1}(k[A \cup B]) \\ \rightarrow H_m^k(k[A \cap B]) \rightarrow H_m^k(k[A] + k[B]) \rightarrow H_m^k(k[A \cup B]) \rightarrow \dots$$

from [BS98, chapter 3] to infer the necessary vanishing.

□

**2.2. Flat families of projective varieties.** We record a proposition, seemingly well-known to the experts, concerning the two-way flow of information between special and generic fibers in a flat family.

**Proposition 2.** *Let  $F \subseteq \mathbb{P}^n \times S$  be a flat family over  $S$  of projective schemes. Assume that the base  $S$  is irreducible and normal.*

- (1) *If the generic fiber is nonempty, then each special fiber is also nonempty.*
- (2) *If a special fiber satisfies Serre's condition  $S_k$ , then the generic fiber does too.*
- (3) *If the reduction of the generic fiber of  $F$  is equidimensional, then the reduction of any fiber is, and the dimensions match.*
- (4) *If the reduction of the generic fiber of  $F$  is connected in codimension 1, then the reduction of any fiber is.*

*Proof.* The map  $F \rightarrow S$  hits a nonempty open set since the generic fiber is nonempty. Being also proper, the map is onto, so each special fiber is nonempty.

For the second claim, let  $C_\eta \subseteq F_\eta$  be the non- $S_k$ -locus of the generic fiber, and  $C \subseteq F$  its closure to a flat family. Since  $S_k$  is a cohomology-vanishing condition and cohomology groups are semicontinuous,  $C_0 \subseteq F_0$ . By part (1),  $C_0$  empty implies  $C_\eta$  empty.

We now combine the technique of slicing with general planes (perhaps after harmlessly extending the base field coming from the point  $o \in S$ ), and the following version of Zariski's Main Theorem: if a flat family of complete schemes over a normal base has connected generic fiber, then all fibers are connected (see e.g. [C57]).

If the reduction of some fiber has a component of small dimension, we slice with a general plane to replace that component by points. Now the generic fiber is still irreducible (Bertini's theorem) hence connected, but the special fiber is disconnected, contradiction. This proves the third claim.

To prove the fourth claim, slice with a general plane to replace  $F$  with a family of curves. The general fiber of this subfamily is still connected, so the special fiber of this subfamily is connected, hence the original special fiber was connected in codimension 1.  $\square$

There are other contexts where part (1) holds, e.g. the case that  $F$  is a family of  $\mathbb{G}_m$ -invariant subschemes of  $\mathbb{P}^n \times \mathbb{A}^k$ , where  $\mathbb{G}_m$  acts linearly on  $\mathbb{P}^n$  and  $\mathbb{A}^k$  and has only positive weights on  $\mathbb{A}^k$ . With some work, one can extend this proposition (and the Reduced Limit Lemma, which depends on it) to that context.

**2.3. Branchvarieties.** We recall the basic construction from [AK] of a *limit branchvariety*.

A **branchvariety**  $X$  of  $Y$  is a map  $\beta : X \rightarrow Y$  of schemes such that  $\beta$  is finite (proper with finite fibers) and  $X$  is (geometrically) reduced. In particular, any closed reduced subscheme of  $Y$  is a branchvariety of  $Y$ ; the prefix *branch* should be seen as analogous to *sub*. The basic facts we need about branchvarieties are collected in the following:

**Theorem 1.** *Let  $F \rightarrow S$  be a flat family of subschemes of  $Y$ , where  $S$  is a normal one-dimensional base, for example  $\text{Spec}$  of a discrete valuation ring. Assume the fiber  $F_o$  is generically geometrically reduced, and that all other fibers are geometrically reduced. Let  $S^\times := S \setminus o$ , and let  $F^\times$  denote the restriction to  $S^\times$ .*

*Then there exists uniquely a flat family  $\tilde{F} \rightarrow S$  of branchvarieties of  $Y$  extending  $F^\times \rightarrow S^\times$ , and a natural finite map  $\beta_o : \tilde{F}_o \rightarrow F_o$  whose image is the reduction  $(F_o)_{\text{red}}$ . This  $\tilde{F}$  may be constructed as the normalization of  $F$  in the open set  $F^\times$ .*

*This  $\beta_o$  induces a correspondence between the top-dimensional components of  $\tilde{F}_o$  and  $F_o$ , and is generically 1:1 on each top-dimensional component.*

*Proof.* The first two paragraphs are theorem 2.5 and corollary 2.6 of [AK]; the base change usually required in [AK, theorem 2.5] may be omitted by the assumption that  $F_o$  is generically geometrically reduced (so each  $m_i = 1$  in the notation of [AK, theorem 2.5]).

If  $C^\times$  is a component of  $F^\times$  not of top dimension, then its closures  $C, \tilde{C}$  inside  $F, \tilde{F}$  are flat subfamilies, whose special fibers  $C_o, \tilde{C}_o$  are therefore also not of top dimension (by proposition 2). So we can safely remove these components of  $F$  without affecting the top-dimensional components of  $F_o, \tilde{F}_o$ . Hereafter we work with the unions  $F', \tilde{F}'$  of the top-dimensional components of  $F$ ; call this dimension  $d$ .

Then, again by proposition 2, we find that  $\tilde{F}'_o, (F'_o)_{\text{red}}$  are also equidimensional of dimension  $d$ . By the finiteness of  $\beta$ , the image of a  $d$ -dimensional component of  $\tilde{F}'_o$  is again of dimension  $d$ , hence a component of  $F'_o$ . So far we have a function from the set of components of  $\tilde{F}'_o$  (i.e. the top-dimensional components of  $\tilde{F}_o$ ) to the set of components of  $F'_o$  (i.e. the top-dimensional components of  $F_o$ ).

The map  $\beta$  induces a top-degree Chow class  $\beta_*([\tilde{F}_o])$  on  $F_o$ , which we can compute as

$$\beta_*([\tilde{F}_o]) = \sum_{D \subseteq \tilde{F}_o} \beta_*([D]) = \sum_{D \subseteq \tilde{F}_o} [\beta(D)] \deg(D \rightarrow \beta(D)) = \sum_{E \subseteq F_o} [E] \sum_{D \subseteq \tilde{F}_o, \beta(D)=E} \deg(D \rightarrow E)$$

where the sums are over top-dimensional reduced components, and  $[Z]$  denotes the fundamental Chow class of the scheme  $Z$ . However, the Chow class shadow of the much more precise K-class statement [AK, proposition 6.1] tells us that  $\beta_*([\tilde{F}_o]) = [F_o]$ , a fact already used in [K06] in the case that  $F$  is a degeneration to a normal cone.

Finally, the fact that  $F_o$  is generically reduced tells us that its fundamental Chow class is simply

$$[F_o] = \sum_{E \subseteq F_o} 1 \cdot [E],$$

so for each top-dimensional component  $E$  of  $F_o$ , we have

$$\sum_{D \subseteq \tilde{F}_o, \beta(D)=E} \deg(D \rightarrow E) = 1.$$

Hence there is only one component  $D$  mapping to  $E$ , and the degree of the map is 1.  $\square$

In fact the construction in theorem 1 does not require the assumption of generic reducedness of  $F_o$ ; the only modification necessary is a certain ramified base change  $S' \rightarrow S$ . Since we assume generic reducedness in the Reduced Limit Lemma, we didn't state here that only slightly more complicated but much more general result (which can be found in [AK]). We mention, though, that in that more general setup the map induced on the sets of top-dimensional components still exists but may be only surjective (as in example 3 of [K06]).

By the uniqueness of the limit branchvariety, if the limit subscheme is reduced, then it agrees with the limit branchvariety. We now sharpen this to a local statement (that again, does not actually require generic reducedness).

**Lemma 3.** *Assume the setup of theorem 1, and let  $U_o \subseteq F_o$  be an open subset. Then the map  $\beta : \beta^{-1}(U_o) \rightarrow U_o$  is an isomorphism iff  $U_o$  is reduced.*

*Proof.* We may pick an open set  $U \subseteq F$  such that  $U \cap F_o = U_o$ . Then the lemma can be rephrased as “for every  $U \subseteq F$ ...” Now observe that the lemma holds for  $U$  iff it holds for an open cover, so it is enough to handle the case  $U$  affine.

Recall now the construction of  $\tilde{F}$ : it is the normalization of  $F$  in the open set  $F \setminus F_o$ . Since normalization commutes with localization to open sets, we see that  $\beta^{-1}(U)$  is the normalization of  $U$  in the open set  $U \setminus U_o$ .

Part (1) of [AK, lemma 2.1] now says that  $U_o$  reduced implies that  $\beta : \beta^{-1}(U) \rightarrow U$  is an isomorphism (and in particular, induces an isomorphism of the fibers over  $o$ ).

Part (2) of [AK, lemma 2.1] only says that  $U_o$  nonreduced implies that after some base change, which can change the normalization, does  $\beta : \beta^{-1}(U') \rightarrow U'$  fail to be an isomorphism. In the case at hand, since  $\beta^{-1}(U_o)$  is reduced, as in [AK, corollary 2.6] the base change can only extend the residue field, so the map is already not an isomorphism.  $\square$

This lemma gives a way to show  $F_o$  is reduced without studying  $F_o$  directly; instead we may show that  $\beta_o : \tilde{F}_o \rightarrow (F_o)_{\text{red}}$  is an isomorphism.

One of the very few surprises in moving beyond subvarieties of projective space to branchvarieties is the failure of some Bertini theorems in characteristic  $p$ : for example the

Frobenius map  $\mathbb{P}_{\mathbb{F}_p}^1 \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$  is a branchvariety whose every hyperplane section is nonreduced. In [AK, assumption 7.2] we got around this by assuming the characteristic was 0 or large enough, but we take a different tack here:

**Lemma 4.** *Let  $\beta : X \rightarrow \mathbb{P}^n$  be a branchvariety, defined over a field, that is birational on each component. Then for a general plane  $P \subseteq \mathbb{P}^n$  (which may require extending the field), the “plane section”  $\beta^{-1}(P) \subseteq X$  is reduced, and itself a branchvariety of  $\mathbb{P}^n$  that is generically 1:1 on each component.*

*Proof.* We may assume that  $P$  is a hyperplane, as we can then use induction.

Since the map  $X \rightarrow \beta(X)$  is birational on each component, it is unramified. So by [J83, Thm. 6.3 (3)], a generic plane section  $\beta^{-1}(P)$  of it is again geometrically reduced.

A proper map  $C \rightarrow D$  of irreducible varieties is generically 1:1 if *some* fiber is a (reduced) point; then the set of  $d \in D$  for which the fiber is a point is open in  $D$ . Since  $P$  intersects this open locus in each component of  $\beta(X)$ , we see that  $\beta^{-1}(P) \rightarrow P \cap \beta(X)$  is again generically 1:1 on each component.  $\square$

**2.4. Geometric vertex decompositions.** We described the setup,  $F_1 = \overline{X} \subseteq H \times \mathbb{P}^1$  degenerating to  $F_0$ , in section 1.2. We now collect (and slightly refine) the results we will need from [KMY07], which partially describe  $F_0$ .

**Theorem 2.** *Let  $X$  be a closed subscheme of  $H \times L$ , where  $H$  is a hyperplane and  $L$  is a line, and let  $\overline{X}$  be its closure in  $H \times \mathbb{P}^1$ . Let  $\Pi \subseteq H$  be the image of  $\overline{X}$  under projection to  $H$ , and define  $\Lambda \subseteq H$  by  $\Lambda \times \{\infty\} := \overline{X} \cap (H \times \{\infty\})$ . Consider the family*

$$F := \overline{\{(h, \ell, z) : (h, z^{-1}\ell) \in \overline{X}\}} \subseteq H \times \mathbb{P}^1 \times \mathbb{A}^1$$

automatically flat over the  $\mathbb{A}^1$  factor. Then as sets,

$$F_0 = (\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{P}^1)$$

and the two agree as schemes away from  $\Pi \times \{0\}$ .

If  $X$  is irreducible and the projection  $X \rightarrow \Pi$  is generically 1:1, then  $F_0$  is generically reduced along  $\Pi$ .

All of this holds if  $H$  is not a vector space, but is merely quasiprojective.

*Proof.* This will be a slight variation of [KMY07, theorem 2.2], in turn based on the algebra from [KMY07, theorem 2.1], which uses coordinates  $\{x_1, \dots, x_n\}, \{y\}$  on  $H, L$ . Let  $I$  be the ideal defining  $X$ . That theorem makes use of a Gröbner basis  $\{y^{d_i} q_i + r_i \mid i = 1 \dots m\}$  of  $I$ , with respect to a term order that picks out a term from the initial  $y$ -form  $y^{d_i} q_i$  of  $y^{d_i} q_i + r_i$ . Theorem 2.1 also defines the ideals

$$I' = \langle y^{d_i} q_i \mid i = 1, \dots, m \rangle, \quad C = \langle q_i \mid i = 1, \dots, m \rangle, \quad P = \langle q_i \mid d_i = 0 \rangle + \langle y \rangle$$

For our first step, we introduce a coordinate  $y'$ , the denominator coordinate on  $\mathbb{P}^1$ . Then we homogenize the generators of  $I$  in  $\{y, y'\}$ , meaning that each term in each  $r_i$  is multiplied by the right power of the new  $y'$  to make the generator  $y^{d_i} q_i + r_i$  homogeneous in  $\{y, y'\}$ . This is the algebraic counterpart of defining  $\overline{X}$  as the closure of  $X$ . Call this  $\{y, y'\}$ -homogeneous ideal  $I_h$ .

Then  $\infty \in \mathbb{P}^1$  is defined by the equation  $y' = 0$ , so  $\Lambda \times \{\infty\} := \overline{X} \cap (H \times \{\infty\})$  is defined by the ideal  $I_h + \langle y' \rangle$ . Projecting to  $H$  amounts to inverting  $y$  and dropping the variables

$y, y'$ , which gives us the ideal  $C$ . If we reintroduce  $y, y'$  as free variables, we get the ideal defining  $\Lambda \times \mathbb{LP}^1$ .

As was observed in [KMY07, theorem 2.2], the limit  $F_0$  is defined by the ideal  $I'$ . Upon inverting  $y$ , the ideals  $I'$  and  $C$  coincide, which is the statement that  $F_0$  and  $\Lambda \times \mathbb{LP}^1$  agree (as schemes) away from  $H \times \{0\}$ .

We can study  $F_0$  away from  $H \times \{\infty\}$  by passing to  $y' = 1$ ; this recovers the affine situation in [KMY07, theorem 2.2], which tells that

$$F_0 \setminus (H \times \{\infty\}) = (\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times L) \quad (\text{not } \mathbb{LP}^1)$$

as sets.

If  $X$  is irreducible and the map  $\bar{X} \rightarrow \Pi$  is generically 1:1, then it is a degree 1 map, and from [KMY07, theorem 2.5] we learn that  $F_0 \setminus (H \times \{\infty\})$  is generically reduced along  $\Pi \times \{0\}$ . Then the same statement holds for  $F_0$ .

**$H$  quasiprojective rather than linear.** The stated result makes sense for  $H$  an arbitrary scheme, not just a vector space, and it is easy to see that

- (1) if the result holds for a scheme  $H$ , and  $H'$  is a subscheme with  $H \supseteq H' \supseteq \Pi$ , then the result holds for  $H'$
- (2) if the result holds for each patch in an open cover of  $H$ , then it holds for  $H$ .

By (1) one can reduce to the case that  $H$  is projective space, and by (2) one can reduce to the already treated case that  $H$  is affine space.  $\square$

The quasiprojectivity assumption seems very unlikely to be necessary. We did not pursue its removal for two reasons: to do so would involve extending the theory of Gröbner bases beyond polynomial rings (or replacing the argument altogether), and our application in section 4 only uses  $H$  linear anyway.

### 3. PROOFS

**3.1. Preliminaries.** We start with a lemma about gluing schemes together along closed subschemes.

**Lemma 5.** *Let  $A, B, X$  be schemes with a map  $A \coprod B \rightarrow X$  such that  $A \rightarrow X, B \rightarrow X$  are embeddings; hence we can identify  $A, B$  with their images in  $X$ . Let  $C$  be the intersection of  $A$  and  $B$  in  $X$ . Then  $X$  (plus the inclusions  $A, B \rightarrow X$ ) is determined up to unique isomorphism by  $C \subseteq A, B$ .*

*Moreover, if the map factors as  $A \coprod B \rightarrow X' \rightarrow X$ , with  $C' = A \cap_{X'} B$ , then  $C' \subseteq C$ , with equality iff the map  $X' \rightarrow X$  is an isomorphism.*

*Proof.* If  $U \subseteq X$  is open, then  $(A \cap U) \coprod (B \cap U) \rightarrow U$  satisfies the same conditions. Conversely, if the statement holds for each  $U$  in an open cover of  $X$ , then it holds for  $X$ . So we can restrict to the case  $A, B, X$  affine, with  $X = \text{Spec } R$ .

Let  $I_A, I_B, I_C$  be the ideals defining  $A, B, C$ . Then  $I_C = I_A + I_B$  by definition. The condition  $X = A \cup B$  says that  $I_A \cap I_B = 0$ . Then  $R$  is the inverse limit of  $R/I_A, R/I_B \rightarrow R/I_C$ , and hence determined up to unique isomorphism by  $C \subseteq A, B$ .

For the second claim, consider the diagram  $A, B \rightarrow X' \rightarrow X$ . Then the pullback  $C'$  of  $A, B \rightarrow X'$  automatically maps to the pullback  $C$  of  $A, B \rightarrow X$ , and since the inclusion  $C' \rightarrow A$  factors as  $C' \rightarrow C \rightarrow A$ , the map  $C' \rightarrow C$  is an inclusion. By the first claim,  $X, X'$

determine and are determined by the subschemes  $C, C'$  of  $A$  and  $B$ , so the map  $X' \rightarrow X$  is an isomorphism iff the inclusion  $C' \hookrightarrow C$  is an isomorphism.  $\square$

The next lemma will be our source of normality for a generic fiber. We take a moment to recall the difference between the *generic* fiber of a family over an irreducible base  $S$ , which is the fiber over the generic point of  $S$ , and a *general* fiber, whose definition only makes sense if  $S$  has enough closed points to have “general” ones. In particular,  $S$  should not be local, and should typically be defined over an infinite field.

A general fiber of  $\mathbb{A}^1 \rightarrow \mathbb{A}^1, z \mapsto z^2$  is reducible, but the generic fiber is the generic point of the source  $F = \mathbb{A}^1$ , so irreducible. A tighter analogue is provided by the **geometric generic fiber** of  $F \rightarrow S$ , made by base-changing  $F$  using the algebraic closure of the function field of  $S$ . In particular, while the general fibers and the geometric generic fiber behave well under many base changes, the generic fiber can go from irreducible to reducible.

For a reduced complete (though possibly disconnected) curve  $C$  with at worst nodal singularities, let  $\Gamma(C)$  denote its **graph of components** (as in e.g. [OS79]), with vertex set the set of components of  $C$ , and edge set the set of nodes of  $C$ . There may be multiple edges between two vertices, and a singular component gives a vertex with self-edges. The graph is connected iff the curve itself is.

**Lemma 6.** *Let  $F \rightarrow S$  be a flat family of at-worst-nodal geometrically reduced curves over an irreducible normal base. Assume that  $F_{\text{sing}} \rightarrow S$  is proper, e.g. if  $F$  itself is. Then there is a natural injection  $\Sigma : \text{edges}(\Gamma(F_\eta)) \rightarrow \text{edges}(\Gamma(F_o))$  from the nodes of the generic fiber to the nodes of the special fiber.*

*Now assume that  $F \rightarrow S$  itself is proper. If the generic fiber is connected, then  $\Gamma(F_o)$  is connected. If the geometric generic fiber is irreducible, then  $\Gamma(F_o)$  remains connected even when the edges in the image of  $\Sigma$  are removed.*

*Proof.* The finite map  $F_{\text{sing}} \rightarrow S$  may be ramified; perform base changes around the special fiber to make it unramified. By the assumptions (used here only) that the geometric generic fiber is irreducible and the special fiber is geometrically reduced, after this finite base change the generic fiber will stay irreducible (if it was) and the special fiber will stay reduced.

Given a node  $N_\eta$  in the generic fiber, take its closure in  $F$  to get a subfamily  $N$  lying in the singular locus  $F_{\text{sing}}$ . Define  $\Sigma(N_\eta) := N_o \in (F_o)_{\text{sing}}$ , which exists by the assumed properness of  $F_{\text{sing}} \rightarrow S$ . This map  $\Sigma$  is an injection, since  $N \cap N' \neq \emptyset$  implies  $(N \cup N')_\eta$  is a fat point sitting inside  $(F_o)_{\text{sing}}$ , but that is reduced.

If the generic fiber is connected, then by proposition 2 the special fiber is too, making its graph connected. In the rest we assume that the generic fiber (after the above base change) is irreducible, which is implied by the geometric generic fiber being irreducible.

By assumption  $N \rightarrow S$  is unramified. A formal neighborhood of  $N$  inside  $F'$  is essentially a deformation over  $S$  of the singularity  $\{xy = 0\}$ . It is easy to compute the universal deformation  $\{xy = t\}$  of this formal singularity, and show that the formal neighborhood

of  $N$  is a trivial family over  $S$  (for any  $t \neq 0$ , the deformation would smooth entirely). So if we blow up  $F$  along  $N$ , it simply detaches that node in each fiber.<sup>1</sup>

We now blow up  $F$  along every subfamily  $N$  coming from a node of  $F_\eta$ . (Note that no two intersect, or else the singular locus of  $F_\eta$  would be nonreduced where two collided, but it is reduced. So there is no worry about the order in which they are blown up.) The generic fiber stays irreducible under these blowings-up, hence connected, so by proposition 2 (which requires the normal base) the special fiber and its graph stay connected. Its new graph is the old one  $\Gamma(F_\eta)$  with the edges in the image of  $\Sigma$  removed.  $\square$

One easy corollary of this is that if  $F$  is a family of projective curves and  $\Gamma(F_\eta)$  is a tree, and the geometric generic fiber is irreducible, it can have no nodes and must be normal. There is another proof of this, explained to us by Johan de Jong and Valery Alexeev. The Jacobian of an at-worst-nodal curve is an abelian variety iff its graph is a tree (see [OS79, Proposition 10.2]). The locus of abelian varieties is open in the Picard scheme, so the condition of having an abelian variety as one's Jacobian is an open condition on the fibers. Hence  $\Gamma(F_\eta)$  is also a tree. By the assumption of irreducibility, that graph has only one vertex, and by treeness, no self-edges. So the generic fiber is one normal component, QED.

We now prove a higher-dimensional version of that corollary, for which we didn't see a Jacobian-based proof.

**Lemma 7.** *Let  $F \rightarrow S$  be a flat family of reduced projective schemes over a normal irreducible base. Assume that the geometric generic fiber is irreducible.*

*Assume that a special fiber  $F_\eta$  has only normal components  $C_1, \dots, C_n$ , where for each  $i$  there exists  $J(i) < i$  such that*

$$C_i \cap (C_1 \cup \dots \cup C_{i-1}) = C_i \cap C_{J(i)}$$

*and this intersection  $C_i \cap C_{J(i)}$  is irreducible and generically reduced. Finally, assume that  $C_i \cap C_j \cap C_k$  has dimension at most  $\dim F_\eta - 2$  for  $i, j, k$  distinct.*

*Then the generic fiber is normal.*

*Proof.* Parallelling the case of curves, we define a graph  $\Gamma$  whose vertices are  $\{1, \dots, n\}$  and with an edge between  $i$  and  $j$  iff  $\dim(C_i \cap C_j) = \dim F_\eta - 1$ . Then since the geometric generic fiber is irreducible, by proposition 2 the special fiber is connected in codimension 1, making this graph  $\Gamma$  connected. The conditions on the intersections, that when listing the vertices in order each attaches to a unique previous one, imply that  $\Gamma$  is a tree. (The converse is not quite true – imagine  $\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$  where the first and third component meet in two points.)

To use Serre's criterion, we must show the generic fiber  $F_\eta$  is  $S_2$  and  $R_1$ . We prove  $C_1 \cup \dots \cup C_i$  is  $S_2$  by induction on  $i$  (the  $i = 1$  case being trivial):

$$C_1 \cup \dots \cup C_i = (C_1 \cup \dots \cup C_{i-1}) \cup_{C_i \cap C_{J(i)}} C_i$$

where the right-hand-side is the union of two  $S_2$  schemes along a reduced scheme. Then proposition 1 says that this scheme is  $S_2$ . For  $i = n$ , we learn that the special fiber is  $S_2$ , so by proposition 2 the geometric generic fiber is  $S_2$ .

<sup>1</sup>In general, blowing up a flat family along a flat subfamily does not commute with passage to fibers. For example, if we blow up the family  $\{(x, y, z) : xy = z^2\}$ ,  $(x, y, z) \mapsto z$  along the section  $\{(z, z, z)\}$ , the  $z = 0$  fiber  $\{xy = 0\}$  acquires a new  $\mathbb{P}^1$  component connecting the now-disjoint axes  $x = 0, y = 0$ .

**Slicing down to a family of curves.** Pick a plane  $P$  in general position with respect to the special fiber and the generic fiber (by extending the base field if necessary), of complementary dimension to  $F_o$  plus one. To show that the generic fiber is  $R_1$ , we want to show that its intersection with  $P$  is  $R_1$ , i.e. that it is a normal curve. The precise generality conditions we want are that

- each  $P \cap C_i$  is a normal curve,
- each  $P \cap C_i \cap C_{J(i)}$  is a set of reduced points,
- each  $P \cap C_i \cap C_j \cap C_k$ , for  $i, j, k$  distinct, is empty, and
- $P$ 's intersection with the geometric generic fiber is a reduced and irreducible curve.

Let  $F'$  be the family of curves given by intersecting every fiber with  $P$ , and  $C'_i = P \cap C_i$  the components of  $F'_o$ . It is worth noting that  $F'$  does *not* satisfy one of the conditions we required of  $F$ , namely that  $C_i \cap C_{J(i)}$  is irreducible. Rather, the corresponding intersection  $C'_i \cap C'_{J(i)}$  is a finite set of points.

The special fiber  $F'_o$  is a union of normal curves  $\{C'_i\}$ . Each  $C'_i \cap C'_j = P \cap C_i \cap C_j$ ,  $j < i$ , is only nonempty if  $j = J(i)$ , in which case it is reduced:  $F'_o$  has only ordinary double points. The graph  $\Gamma(F'_o)$  is almost the same as the graph  $\Gamma$  constructed above; the only difference is that two connected vertices will, as explained in the last paragraph, usually have *many* edges between them.

**Using nodes in  $F'_n$  to disconnect  $F'_o$ .** If  $F'_n$  is regular, we are done. Otherwise we may pick a singular point (a node) in  $F'_n$ , and take its closure  $N$  in  $F'$ . Then  $N_o$  is a node in  $F'_o$ , so a point in  $C'_i \cap C'_{J(i)}$  for some unique  $i$ . This  $N$  and  $i$  are fixed hereafter.

We now claim that for *every* point (node)  $p \in C'_i \cap C'_{J(i)}$ , there exists a section  $N^p \subseteq F'_{\text{sing}}$  of  $F' \rightarrow S$  with  $N^p_o$  the desired node. This follows from the assumed irreducibility of  $C_i \cap C_{J(i)}$ , as follows. By varying the general plane  $P$  we can vary the intersection  $C'_i \cap C'_{J(i)}$ , and the nodes for which there do, resp. don't, exist such sections sweeps out an open set in  $C_i \cap C_{J(i)}$ . By the irreducibility, one of these two open sets is empty; since we used  $N$  to choose  $i$  we know it is the set of nodes for which there don't exist such sections.

This says that the map  $\Sigma$  from lemma 6 surjects onto the edges connecting  $C'_i, C'_{J(i)}$  in the graph of  $F'_o$ . Removing those edges disconnects the graph, counter to the result of lemma 6. This contradiction traces back to our assuming that  $F'_n$  was not regular.  $\square$

Our last technical lemma contains a couple of simple observations about the families in theorem 2 concerning geometric vertex decompositions.

**Lemma 8.** *For any  $Y \subseteq H \times \mathbb{L}\mathbb{P}^1$  as in theorem 2, let  $F(Y)$  denote the flat family constructed there. Let  $\bar{X} \subseteq H \times \mathbb{L}\mathbb{P}^1$  be the closure of  $X \subseteq H \times L$ . Then*

- (1)  $X$  is nonempty iff  $F(\bar{X})_o$  is nonempty.
- (2) If  $\bar{X}$  is an irreducible curve satisfying the conditions of theorem 2, and its projection  $\Pi \subseteq H$  is normal, then  $\bar{X}$  is itself normal.

*Proof.* (1) If  $X$  is nonempty, then  $\bar{X}$  and its projection  $\Pi$  are nonempty, and  $F(\bar{X})_o \supseteq \Pi \times \{0\}$  so it too is nonempty. The converse is obvious.

- (2) The argument from lemma 6 must be modified slightly, because the map  $F(\bar{X}) \rightarrow \mathbb{A}^1$  is not proper, and if one simply compactifies one may add singularities that are worse than nodes. The key observations are that

- Any singularity  $N \in \bar{X}_{\text{sing}}$  gives a subfamily  $F(N) \subseteq F(\bar{X})_{\text{sing}}$  that is proper over  $\mathbb{A}^1$ . Since  $\Pi$  is normal,  $F(\bar{X})_o = (\Pi \times \{0\}) \cup (\Lambda \times \mathbb{P}^1)$  is nodal, so  $\bar{X}$  is at worst nodal.
- The new points in the compactification attach to only one component of  $F(\bar{X})_o$ , namely  $\Pi \times \{0\}$ , so aren't relevant in studying connectedness.

If  $N \in \bar{X}_{\text{sing}}$ , then  $F(N)_o$  is necessarily one of the nodes  $\Lambda \times \{0\}$  of  $(\Pi \times \{0\}) \cup (\Lambda \times \mathbb{P}^1)$ , so as before the formal neighborhood of  $F(N)$  inside  $F(\bar{X})$  is a trivial deformation of a node. Now compactify, blow up  $F(\bar{X})$  along  $F(N)$ , and as before get an impossible family of projective curves whose generic fiber is irreducible but whose special fiber is disconnected.

□

### 3.2. Proofs of the main lemmas.

*Proof of the Reduced Limit Lemma.* Via base change, we can reduce to the case that  $S$  is the germ of a regular 1-dimensional scheme, e.g. the Spec of a discrete valuation ring  $D$ . Then  $S$  has one closed point and one open point.

By theorem 1, the family  $F$  is dominated by a family  $\tilde{F}$  of branchvarieties, agreeing over  $S \setminus o$ . Note that  $\tilde{F}_o, F_o$  are each equidimensional of dimension  $d$  and connected in codimension 1, by proposition 2. The branchvariety  $\tilde{F}_o \rightarrow F_o$  appears a priori to depend on the curve chosen in the first step, but this will not affect the argument (which will in any case establish that  $\tilde{F}_o = F_o$ ).

**The components of  $\tilde{F}_o$ .** Let  $\beta_o : \tilde{F}_o \rightarrow F_o$  be the induced map on special fibers, with image  $(F_o)_{\text{red}} = A_1 \cup \dots \cup A_k$ . We will now show that  $\tilde{F}_o$  has exactly the same components, though a priori they may be glued together differently.

By the latter conclusion of theorem 1, the components of  $\tilde{F}_o$  can be labeled  $\tilde{A}_1, \dots, \tilde{A}_k$ , with  $\beta_o(\tilde{A}_i) = A_i$ , and each map  $\tilde{A}_i \rightarrow A_i$  is degree 1 and finite. Now we make use of the assumption that the  $\{A_i\}$  are normal, which lets us infer that each map  $\tilde{A}_i \rightarrow A_i$  is an isomorphism. So  $(F_o)_{\text{red}}, \tilde{F}_o$  have the same components, as claimed. Hereafter we identify the components of  $\tilde{F}_o$  with the  $\{A_i\}$ .

**Showing  $F_o$  is reduced.** We now split into the three cases of the lemma:  $k = 1$ ,  $k = 2$ , and general  $k$ . In each case, rather than dealing with  $F_o$  directly, the idea is to show that the map  $\beta_o : \tilde{F}_o \rightarrow (F_o)_{\text{red}}$  is an isomorphism. Then lemma 3 lets us infer indirectly that  $F_o$  is reduced.

*If  $k = 1$ .* Then  $\tilde{F}_o = A_1 = (F_o)_{\text{red}}$ . By lemma 3,  $F_o$  is reduced, and obviously normal (since  $A_1$  was assumed so).

*If  $k = 2$ , and  $A_1 \cap A_2 \subseteq (F_o)_{\text{red}}$  is reduced and irreducible.* Consider the diagrams  $A_1, A_2 \rightarrow (F_o)_{\text{red}}$  and  $A_1, A_2 \rightarrow \tilde{F}_o$ , and denote their pullbacks (which are just the intersections) by  $C, C'$ . Then by lemma 5, there is an inclusion  $C' \hookrightarrow C$ , and our goal is to show their equality. Since  $\tilde{F}_o$  is connected in codimension 1,  $\dim C' = \dim F_o - 1 = \dim C$ .

Now we use the assumption that  $C$  is reduced and irreducible to infer  $C' = C$ . Hence by lemma 5 the map  $\tilde{F}_o \rightarrow (F_o)_{\text{red}}$  is an isomorphism, so lemma 3 tells us  $F_o$  is reduced.

General  $k$ ,  $F_o$  reduced through codimension 1, and the shelling condition. Let  $\tilde{F}_o^i, (F_o)_{\text{red}}^i$  denote the unions of the images of  $A_1, \dots, A_i$  in  $\tilde{F}_o, (F_o)_{\text{red}}$ . So we have a map  $\tilde{F}_o^i \rightarrow (F_o)_{\text{red}}^i$  for each  $i$ . Assume that  $\tilde{F}_o^j \rightarrow (F_o)_{\text{red}}^j$  is an isomorphism for all  $j < i$ ; we will use this to prove it for  $j = i$ . The base case  $i = 1$  is trivial, as the map is  $A_1 \rightarrow A_1$ .

This proof is very similar to the one just given, except that we work with the pullback diagrams of  $\tilde{F}_o^{i-1}, A_i \rightarrow (F_o)_{\text{red}}^i$  and  $\tilde{F}_o^{i-1}, A_i \rightarrow \tilde{F}_o^i$ . Again call the pullbacks  $C, C'$ , and again we have  $C' \subseteq C$ . By assumption  $C$  is reduced, and equidimensional of dimension  $\dim F_o - 1$ . So either  $C' = C$ , or  $C'$  does not contain some component  $D$  of  $C$ .

If not, then the map  $\tilde{F}_o^i \rightarrow (F_o)_{\text{red}}^i$  is generically 2:1 over  $D$  — once from  $\tilde{F}_o^{i-1}$ , once from  $A_i$ . But then by lemma 3,  $F_o$  would not be generically reduced along  $D$ .

Hence  $C' = C$ , so  $\tilde{F}_o^i \rightarrow (F_o)_{\text{red}}^i$  is an isomorphism by lemma 5. When  $i = k$ , we learn that  $\tilde{F}_o \rightarrow (F_o)_{\text{red}}$  is an isomorphism, so  $F_o$  is reduced by lemma 3.

**Showing the generic fiber is  $S_2$ .** In case (1) we saw that the special fiber is normal, hence  $S_2$ , so the generic fiber is  $S_2$  by proposition 1. We now treat cases (2) and (3) together.

First we show the (reduced!) special fiber  $F_o$  is  $S_2$ . This is by induction on  $k$ , using

$$F_o^i = F_o^{i-1} \cup_{F_o^{i-1} \cap A_i} A_i \quad \text{for } i > 0, \text{ and } F_o^0 := \emptyset \quad (2)$$

where  $F_o^i = \tilde{F}_o^i = (F_o)_{\text{red}}^i$ . By assumption,  $F_o^{i-1} \cap A_i$  is reduced so  $S_1$ ,  $A_i$  is normal so  $S_2$ , and  $F_o^{i-1}$  is  $S_2$  by induction on  $i$ . Their union  $F_o^i$  is then  $S_2$  by proposition 1. At  $i = k$  we find out  $F_o$  is  $S_2$ .

Hence by proposition 2, the generic fiber is also  $S_2$ .

**(In cases (1) and (2)) The generic fiber is normal.** In case (1), if the abnormal locus in the generic fiber is nonempty, its closure will give a subfamily whose special fiber will be nonempty and lie in the abnormal locus of  $F_o$ , contradiction. (We could also just invoke the local result of [H58].)

Case (2) is exactly the situation of lemma 7 with either order on the two components, and there are no triple intersections to consider.  $\square$

In cases (1) and (2) we proved the generic fiber to be normal. This conclusion need not hold in case (3): consider a nodal plane cubic degenerating to a union of a line and a conic.

Case (2) can be considerably generalized along the lines of the intersection conditions in lemma 7. The proof is not any more difficult, but we omitted it as we know no natural examples not already covered by case (2).

*Proof of lemma 1.* Use equation (2) above, proposition 1, and induction, exactly as was done in the proof above to prove  $F_o$  was  $S_2$ .  $\square$

*Proof of the Geometric Vertex Decomposition Lemma.* The proof is very close to that of the Reduced Limit Lemma, and we use the same notation  $S, F_o, \tilde{F}_o, \beta_o : \tilde{F}_o \rightarrow F_o$ .

By theorem 2, we have the containment of schemes

$$F_o \supseteq (\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{P}^1)$$

and the difference is supported on  $\Pi \times \{0\}$ , in codimension  $\geq 1$ . (Note that  $\Pi$  is automatically irreducible, being the image of  $\bar{X}$ .) In particular  $F_o$  is generically reduced.

**The component A.** Hence by theorem 1 there is a component  $A$  of  $\tilde{F}_o$  mapping to  $\Pi \times \{0\}$ , and the map is degree 1. Having assumed  $\Pi$  to be normal, we infer that this finite map  $A \rightarrow \Pi$  is an isomorphism.

**The union of components B.** Consider now the preimage  $\beta_o^{-1}(F_o \setminus (\Pi \times \{0\})) \subseteq \tilde{F}_o$ . Since  $F_o$  is reduced on  $F_o \setminus (\Pi \times \{0\}) = \Lambda \times (L\mathbb{P}^1 \setminus \{0\})$  by the assumption that  $\Lambda$  is reduced, using lemma 3 we can see that the map

$$\beta_o : \beta_o^{-1}(F_o \setminus (\Pi \times \{0\})) \rightarrow F_o \setminus (\Pi \times \{0\})$$

is an isomorphism. Let  $B \subseteq \tilde{F}_o$  denote the closure of this preimage.

Since a component of  $\tilde{F}_o$  maps either to  $\Pi \times \{0\}$  or to the closure of its complement, we see that  $\tilde{F}_o = A \cup B$ . The main difference between this situation and case (2) of the Reduced Limit Lemma is that  $B$  is usually not irreducible.

To continue following the argument in the Reduced Limit Lemma, we will need to determine  $B$ , and show that  $A, B$  are glued together the same way in  $\tilde{F}_o$  as in  $(F_o)_{\text{red}}$ .

The image of  $B$  is the closure of  $F_o \setminus (\Pi \times \{0\})$ , namely  $\Lambda \times L\mathbb{P}^1$ . So the map  $B \rightarrow \Lambda \times L\mathbb{P}^1$  is finite, degree 1, and an isomorphism away from  $\Lambda \times \{0\}$ . This forces it to be an isomorphism everywhere. (This uses the normality of  $L\mathbb{P}^1$  rather than any condition on  $\Lambda$ .) If we use this to identify  $B$  with  $\Lambda \times L\mathbb{P}^1$ , we can decompose

$$\tilde{F}_o = (\Pi \times \{0\}) \cup_{C' \times \{0\}} (\Lambda \times L\mathbb{P}^1)$$

where  $C'$  is defined by the intersection.

In  $(F_o)_{\text{red}}$ , namely  $(\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times L\mathbb{P}^1)$ , the intersection  $(\Pi \times \{0\}) \cap (\Lambda \times L\mathbb{P}^1)$  is  $\Lambda \times \{0\}$  as a scheme. In particular this intersection is equidimensional (being Cartier in  $\bar{X}$ ) and reduced (by assumption). By lemma 5,  $C' \subseteq \Lambda$ . It remains to show that  $C'$  contains general points from each component of  $\Lambda$ , and thereby learn  $C' \times \{0\} = \Lambda \times \{0\}$ . For this we can safely extend the base field (from the point  $o \in S$ ) to its algebraic closure.

**Slicing down to the 1-dimensional case.** Let  $P \subseteq H$  be a plane in general position (in particular, not necessarily through  $\vec{0}$ ) with respect to  $\Pi$  and  $\Lambda$ , whose intersection with  $\Lambda$  is 0-dimensional. Then

$$\begin{aligned} (F_o)_{\text{red}} \cap (P \times L\mathbb{P}^1) &= ((\Pi \cap P) \times \{0\}) \cup_{(\Lambda \cap P) \times \{0\}} ((\Lambda \cap P) \times L\mathbb{P}^1) \\ \beta_o^{-1}(P \times L\mathbb{P}^1) &= ((\Pi \cap P) \times \{0\}) \cup_{(C' \cap P) \times \{0\}} ((\Lambda \cap P) \times L\mathbb{P}^1) \end{aligned}$$

Now we make our only use of lemma 4, to say that  $\beta_o^{-1}(P \times L\mathbb{P}^1)$  is reduced and that  $\beta_o^{-1}(P \times L\mathbb{P}^1 \times S)$  is a flat family of branchvarieties. Since its generic fiber is irreducible (by Bertini's theorem), proposition 2 says that  $\beta_o^{-1}(P \times L\mathbb{P}^1)$  is connected.

In  $\beta_o^{-1}(P \times L\mathbb{P}^1) = ((\Pi \cap P) \times \{0\}) \cup_{(C' \cap P) \times \{0\}} ((\Lambda \cap P) \times L\mathbb{P}^1)$ , the first term  $(\Pi \cap P) \times \{0\}$  is a normal affine curve, and  $(\Lambda \cap P) \times L\mathbb{P}^1$  is a disjoint union of  $\mathbb{P}^1$ 's. For each point  $\lambda$  in the finite set  $\Lambda \cap P$ , the only possible point of intersection of  $\lambda \times L\mathbb{P}^1$  and any other component of  $\tilde{F}_o$  is  $\lambda \times \{0\}$ . So for  $\beta_o^{-1}(P \times L\mathbb{P}^1)$  to be connected, we must have  $\lambda \in C'$ .

This demonstrates that  $C'$  contains general enough points of  $\Lambda$  to contain all of  $\Lambda$ 's top-dimensional components, which with  $\Lambda$  reduced says that  $C' = \Lambda$ .

Finally, we invoke lemma 5 to infer that the map  $\tilde{F}_o \rightarrow (F_o)_{\text{red}}$  is an isomorphism, then lemma 3 to infer that  $F_o$  is reduced.

**Cohen-Macaulayness.** We now assume  $\Pi$  and  $\Lambda$  are Cohen-Macaulay. Then so is  $\Lambda \times \mathbb{LP}^1$ , so

$$(\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{LP}^1)$$

is a union of two Cohen-Macaulay schemes along a third of codimension 1. Hence  $F_0$  is Cohen-Macaulay by proposition 1.

We would like to claim that  $\bar{X}$  is Cohen-Macaulay by proposition 2, but that proposition assumes projectivity. (Essentially, the problem is that in the nonprojective situation, one can have nonempty families with empty special fibers, and we need a different way to forbid this.) So instead we consider the non-C-M locus  $B_1 \subseteq F_1 = \bar{X}$ , and complete it to a subfamily  $B \subseteq F$  using the same recipe. Since  $B_0$  lies inside the (empty) non-C-M locus of  $F_0$ , it too is empty. By lemma 8, since  $B_0 = \emptyset$ , then  $B_1 = \emptyset$  also.

Finally,  $X$  is Cohen-Macaulay since it is open in  $\bar{X}$ .  $\square$

*Proof of lemma 2.* Let  $C_1 \subseteq \bar{X}_{\text{sing}}$  be a component of the singular locus, and of codimension 1 in  $\bar{X}$ . Let  $C \subseteq F$  be the subfamily constructed by the same recipe as  $F$ . Then  $C_t \subseteq (F_t)_{\text{sing}}$  for all  $t \in \mathbb{A}^1$ .

**Showing**  $C_1 \not\subseteq \Lambda \times \{\infty\}$ . Assume for contradiction that  $C_1 \subseteq \Lambda \times \{\infty\}$ . Then  $C$  is a constant family, so  $C_0 \subseteq (\Lambda \times \{\infty\}) \cap (F_0)_{\text{sing}}$ . By theorem 2

$$F_0 \setminus (H \times \{0\}) = \Lambda \times (\mathbb{LP}^1 \setminus \{0\})$$

so

$$(F_0 \setminus (H \times \{0\}))_{\text{sing}} = (\Lambda \times (\mathbb{LP}^1 \setminus \{0\}))_{\text{sing}} = \Lambda_{\text{sing}} \times (\mathbb{LP}^1 \setminus \{0\}).$$

Hence  $C_0 \subseteq \Lambda_{\text{sing}} \times \{\infty\}$ .

Since  $\Lambda$  is generically reduced (indeed, it was assumed reduced), its singular locus  $\Lambda_{\text{sing}}$  contains no top-dimensional components of  $\Lambda$ . But then its dimension is too small to contain  $C_0$ , contradiction.

**Showing**  $C_1 = D \times \mathbb{LP}^1$ . Extend the base field as usual, so we may slice with a general plane  $P \subseteq H$  such that

- $P \cap \Pi$  is a normal curve, and
- $P \cap \bar{X}$  is irreducible and projects generically  $1 : 1$  to  $P \cap \Pi$ , and
- $P \cap \Lambda$  is reduced and 0-dimensional.

Then

$$\begin{aligned} F_0 \cap (P \times \mathbb{LP}^1) &= ((\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{LP}^1)) \cap (P \times \mathbb{LP}^1) \\ &= ((\Pi \cap P) \times \{0\}) \cup_{(\Lambda \cap P) \times \{0\}} ((\Lambda \cap P) \times \mathbb{LP}^1) \end{aligned}$$

is nodal, so we can apply lemma 8 part (2) to infer that  $\bar{X} \cap (P \times \mathbb{LP}^1)$  is a normal curve.

At this point we assume, for intended contradiction, that  $C$  is *not* of the form  $D \times \mathbb{LP}^1$ . Therefore its projection  $\Pi_C \subseteq H$  is of the same dimension as  $C$ , so  $\Pi_C \cap P$  is a nonempty set of points. Consequently,  $C \cap (P \times \mathbb{LP}^1)$  is a nonempty set of singularities of the curve  $\bar{X} \cap (P \times \mathbb{LP}^1)$ , contradiction.

**Studying**  $D$ . What can we say about the factor  $D$  in  $C_1 = D \times \mathbb{LP}^1$ ? Since

$$\Lambda \times \{\infty\} = \bar{X} \cap (H \times \{\infty\}) \supseteq C_1 \cap (H \times \{\infty\}) = (D \times \mathbb{LP}^1) \cap (H \times \{\infty\}) = D \times \{\infty\},$$

we see  $\Lambda \supseteq D$ . By dimension count,  $D$  is codimension 1 in  $\Lambda$ , and  $C_1 \subseteq \bar{X}_{\text{sing}}$  implies  $D \subseteq \Lambda_{\text{sing}}$ , whose codimension in  $\Lambda$  is at least 1 (since  $\Lambda$  was assumed reduced). Hence  $D$  is a top-dimensional (in particular, non-embedded) component of  $\Lambda_{\text{sing}}$ .

If  $\Lambda$  is normal, then it is  $R_1$  so there can be no such  $D$  (since  $\Lambda_{\text{sing}}$  is of too low dimension) thus no such  $C_1$ , hence  $\bar{X}$  also is  $R_1$ . Also, by the same Mayer-Vietoris argument as in the proof of Cohen-Macaulayness in the previous lemma,  $\Lambda$  and  $\Pi$  being  $S_2$  (since they are normal) implies that  $\bar{X}$  is  $S_2$ . Together, we see that  $\Lambda$  normal implies  $\bar{X}$  is normal.  $\square$

We now give an example showing the criterion in lemma 2 is not automatic. Let

$$\begin{aligned}\Pi = H &= \text{Spec } \mathbb{C}[x, y] \\ \mathbb{L}\mathbb{P}^1 &= \text{Proj } \mathbb{C}[a^{(1)}, b^{(1)}] \quad (\text{superscripts indicating degrees}) \\ \bar{X} &= \text{Proj } \mathbb{C}[x^{(0)}, y^{(0)}, a^{(1)}, b^{(1)}]/\langle ax^2 - (a+b)y^2 \rangle\end{aligned}$$

so

$$\Lambda = \{(x, y, [a, b]) \in \bar{X} : b = 0\} \cong \{(x, y) : x = \pm y\}$$

Then all the other conditions hold:  $\bar{X}$  is irreducible,  $\bar{X} \rightarrow \Pi$  is generically 1:1,  $\Pi$  is normal,  $\Lambda$  is  $S_2$ , but  $\bar{X}$  is not  $R_1$ . According to lemma 2, we may blame this on the  $\mathbb{P}^1$  of singularities along  $\{x = y = 0\}$ , which happens to be the support of the whole singular locus of  $\bar{X}$ .

#### 4. AN APPLICATION TO SCHUBERT VARIETIES

In this section we use the Geometric Vertex Decomposition Lemma and lemma 2 to study the singularities of Schubert varieties in generalized flag manifolds  $G/B$ .

Most of the results here are standard, or at least well-known to the experts, with the exception of lemma 10 and of course the new proof of theorem 3. Since our goal is exactly to provide a new proof, we felt it was worth making the argument largely self-contained, the better to demonstrate that we haven't hidden an old proof somewhere. The exceptions to self-containment are some structure theory of reductive groups and the BGG/Demazure iterative construction of Schubert varieties.

**4.1. Copies of  $H \times \mathbb{L}\mathbb{P}^1$  inside  $G/B$ .** Fix a pinning  $(G, T, W, B, B_-)$  of a reductive Lie group, where  $B$  and  $B_-$  are opposed Borel subgroups with intersection  $T$ . Let  $N_-$  denote the unipotent radical of  $B_-$ , so  $B_- = TN_-$ . Associated to a simple root  $\alpha$  we have the simple reflection  $r_\alpha \in W := N(T)/T$  and the minimal parabolic subgroup  $P_\alpha = \langle B, r_\alpha B r_\alpha \rangle$ . Let  $P_{-\alpha} = \langle B_-, r_\alpha B_- r_\alpha \rangle$  denote the corresponding extension of  $B_-$ .

For example,  $G$  might be the group  $GL_n$  of invertible matrices,  $B$  the upper triangular matrices,  $P_\alpha$  the matrices whose lower triangle vanishes except at the matrix entry  $(j+1, j)$ ,  $B_-$  the lower triangulars, and  $T$  the diagonals. Our interest is in the **generalized flag manifold**  $G/B$ , which is isomorphic in the case  $G = GL_n$  to the space of full flags in the vector space  $\mathbb{A}^n$ . In general, since our interest is in the action on  $G/B$ , there is no harm in replacing  $G$  by its adjoint group  $G/Z(G)$ . Let  $\pi_\alpha : G/B \rightarrow G/P_\alpha$  denote the canonical submersion, a bundle map with fibers  $P_\alpha/B \cong \mathbb{P}^1$ .

It turns out that  $G/B$  is an especially natural venue in which to apply the Geometric Vertex Decomposition Lemma: it contains many open subvarieties of the form  $H \times \mathbb{L}\mathbb{P}^1$ . To locate them we will need a couple of preparatory statements.

**Lemma 9.** (1) Let  $X \subseteq N_-$  be closed,  $T$ -invariant, and nonempty. Then  $X \ni 1$ .  
(2) Let  $v \in W$ , and  $N_1 = N_- \cap vN_-v^{-1}$ ,  $N_2 = N_- \cap vNv^{-1}$ . Then the multiplication maps  $N_1 \times N_2 \rightarrow N_-$ ,  $N_2 \times N_1 \rightarrow N_-$  are isomorphisms of schemes.

*Proof.* If  $\sigma : \mathbb{G}_m \rightarrow T$  is a regular dominant coweight, then

$$\forall n \in N_-, \quad \lim_{t \rightarrow 0} \text{ad } \sigma(t) \cdot n = 1.$$

So  $\exists n \in X$  plus  $X$  invariant under  $T$  (hence under  $\sigma$ ) implies  $1 \in X$ .

We consider the first map in the second claim (the argument is the same for the second map). This map is  $T$ -equivariant with respect to the conjugation action of  $T$  on  $N_-$ ,  $N_1$ ,  $N_2$ . Hence the semialgebraic sets

$$\{(n_1, n_2) \in N_1 \times N_2 : \exists (n'_1, n'_2) \neq (n_1, n_2), n'_1 n'_2 = n_1 n_2\}$$

$$\{n \in N_- : \exists (n_1, n_2) \in N_1 \times N_2, n_1 n_2 = n\}$$

the ramification locus in  $N_-$

are each  $T$ -invariant, and we can apply the first claim to their closures.

The derivative at the identity of this map is the isomorphism  $n_1 \oplus n_2 \rightarrow n_-$ . By the inverse function theorem, the multiplication map is a diffeomorphism near the identity of  $N_1, N_2$ . So these sets cannot have the identity in their closure, and hence must be empty. Therefore this map is a bijective, unramified map between normal varieties, hence an isomorphism.  $\square$

Since the map  $\pi_\alpha : G/B \rightarrow G/P_\alpha$  is a fiber bundle, it can be trivialized over some atlas of the target, and the following proposition specifies one of local trivializations. We will need the fact that  $N_-$  acts on  $G/B$  with a free open dense orbit, called the **big cell**.

**Proposition 3.** Let  $\text{Rad}(P_{-\alpha}) := N_- \cap r_\alpha N_- r_\alpha$ , the unipotent radical of  $P_{-\alpha}$ . Pick a group isomorphism  $F_{-\alpha} : \mathbb{G}_a \rightarrow N_{-\alpha} := N_- \cap r_\alpha N r_\alpha$ . Then the map

$$\mathbb{A}^1 \rightarrow G/B, \quad z \mapsto F_{-\alpha}(z)B/B$$

extends continuously to an embedding  $\bar{F}_{-\alpha} : \mathbb{P}^1 \rightarrow G/B$ , and the  $T$ -equivariant map

$$\begin{aligned} \gamma : \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1 &\rightarrow G/B \\ (n, z) &\mapsto n F_{-\alpha}(z)B/B \end{aligned}$$

is an open immersion, with image  $\pi_\alpha^{-1}(N_- P_\alpha / P_\alpha)$ . The diagram of  $\mathbb{P}^1$ -bundles

$$\begin{array}{ccc} \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1 & \xrightarrow{\gamma} & \pi_\alpha^{-1}(N_- P_\alpha / P_\alpha) \subseteq G/B \\ \downarrow & & \downarrow \pi_\alpha \quad \downarrow \pi_\alpha \\ \text{Rad}(P_{-\alpha}) & \xrightarrow{\cdot P_\alpha / P_\alpha} & N_- P_\alpha / P_\alpha \subseteq G/P_\alpha \end{array}$$

commutes, and the horizontal arrows are isomorphisms, making  $\gamma$  a trivialization of the  $\mathbb{P}^1$ -bundle  $\pi_\alpha$  restricted to the open set  $N_- P_\alpha / P_\alpha \subseteq G/P_\alpha$ .

Let  $X_{r_\alpha}^\circ := N_- r_\alpha B / B \subseteq G/B$ . Then the locally closed subsets

$$\gamma(\text{Rad}(P_{-\alpha}) \times \{\infty\}) = X_{r_\alpha}^\circ, \quad \gamma(\text{Rad}(P_{-\alpha}) \times \{0\}) = r_\alpha X_{r_\alpha}^\circ$$

are sections over  $N_- P_\alpha / P_\alpha$  of this  $\mathbb{P}^1$ -bundle.

*Proof.* The restriction of  $\gamma$  to the open set

$$\text{Rad}(P_{-\alpha}) \times \mathbb{A}^1 \rightarrow G/B$$

factors through  $N_-$  by lemma 9, hence that restriction is an open immersion onto the big cell. We must show that the extension exists and is finite (e.g. injective), with image a normal variety, to conclude that it is an isomorphism onto the image. Being an extension of an injective immersion, it is automatically degree 1.

The extension of

$$\mathbb{A}^1 \rightarrow G/B, \quad z \mapsto F_{-\alpha}(z)B/B$$

to  $\mathbb{P}^1$  exists because  $G/B$  is proper. Since the group  $F_{-\alpha}$  is contained in  $P_\alpha$ , the image of  $\bar{F}_{-\alpha}$  is contained in the fiber  $P_\alpha/B$  of the map  $\pi_\alpha$ , and as they are both 1-dimensional, closed, and reduced the image must equal that fiber. Also, this degree 1 proper map to a normal target must be an isomorphism. We mention that the previously missed point in  $P_\alpha/B$  is  $r_\alpha B$ .

By lemma 9 applied to  $v = r_\alpha$ , the intersection  $\text{Rad}(P_{-\alpha}) \cap F_{-\alpha}$  is trivial. Hence the orbit through the basepoint of  $G/P_\alpha$  is free, and by dimension count, open dense. Call this orbit the **big cell on  $G/P_\alpha$** .

We now claim that  $\gamma$  is an isomorphism of  $\text{Rad}(P_{-\alpha}) \times \mathbb{P}^1$  and  $\pi_\alpha^{-1}(N_- P_\alpha/P_\alpha)$ . Since  $\pi_\alpha$  is  $G$ -equivariant, each element  $n \in \text{Rad}(P_{-\alpha}) \leq G$  permutes the fibers. Because  $\text{Rad}(P_\alpha)$  acts freely on the big cell on  $G/P_\alpha$ , it doesn't preserve any fiber, which shows that  $\gamma$  is injective.

The image of  $\gamma$  is obviously a union of fibers, and composing with  $\pi_\alpha$  the map becomes  $(n, z) \mapsto n P_\alpha B/B$ , whose image doesn't change if we replace  $n \in \text{Rad}(P_{-\alpha})$  by  $n \in \text{Rad}(P_{-\alpha}) F_{-\alpha} = N_-$ . Hence the image of  $\gamma$  is  $\pi_\alpha^{-1}(N_- P_\alpha/P_\alpha)$  as claimed.

Since  $\pi_\alpha \circ \gamma$  is proper, so is  $\gamma$ , hence it is a proper bijective degree 1 map to its normal image, and thus an isomorphism.

Obviously  $\text{Rad}(P_{-\alpha}) \times \{z\}$  is a section of the left-hand bundle for  $z = 0, \infty$  or indeed any  $z \in \mathbb{P}^1$ . We compute the images, using  $N_\alpha = N \cap r_\alpha N_- r_\alpha$ :

$$\begin{aligned} \gamma(\text{Rad}(P_{-\alpha}) \times \{\infty\}) &= \text{Rad}(P_{-\alpha}) r_\alpha B/B = \text{Rad}(P_{-\alpha}) r_\alpha N_\alpha B/B = \text{Rad}(P_{-\alpha}) N_- r_\alpha B/B \\ &= N_- r_\alpha B/B = X_{r_\alpha}^\circ \\ \gamma(\text{Rad}(P_{-\alpha}) \times \{0\}) &= \text{Rad}(P_{-\alpha}) B/B = r_\alpha \text{Rad}(P_{-\alpha}) r_\alpha B/B = r_\alpha X_{r_\alpha}^\circ. \end{aligned}$$

□

The image of  $\bar{F}_{-\alpha}$  is a  $T$ -invariant  $\mathbb{P}^1$  inside  $G/B$ , whose  $T$ -fixed points are  $\{B, r_\alpha B\}$ . One consequence of proposition 3 is that this  $\mathbb{P}^1$  has trivial normal bundle (and enjoys a tubular neighborhood theorem, a rarity in algebraic geometry). This triviality does *not* hold on partial flag manifolds  $G/P$ ; for example the normal bundle to a  $T$ -invariant  $\mathbb{P}^1 \subset \mathbb{P}^2$  is  $\mathcal{O}(1)$ , not trivial. This is the uncommon situation in which  $G/B$  is simpler than  $G/P$ .

**4.2. Schubert varieties and patches.** The **Chevalley-Bruhat decomposition** of the **generalized flag manifold**  $G/B$  is by orbits of  $N_-$ , which are indexed by the Weyl group  $W$ :

$$G/B = \coprod_{w \in W} X_w^\circ, \quad X_w^\circ := N_- w B/B.$$

(Technically,  $w \in W = N_G(T)/T$  should be lifted to an element  $\tilde{w} \in N_G(T)$ , but  $\tilde{w}B$  doesn't depend on this choice, so we don't clutter the notation with it.) A **Schubert variety**  $X_w$  is the closure  $\overline{X_w^\circ} \subseteq G/B$  of a **Schubert cell**  $X_w^\circ$ . The big cell is  $X_1^\circ$ . The **Bruhat order** on  $W$  has  $v \geq w$  if  $vB \in X_w$ . Then each Schubert variety has a Bruhat decomposition  $X_w = \coprod_{v \geq w} X_v^\circ$ .

We summarize what we need of the Bernstein-Gel'fand-Gel'fand/Demazure iterative construction of Schubert varieties.

**Proposition 4.** *Let  $\pi_\alpha$  denote the canonical submersion  $G/B \rightarrow G/P_\alpha$ . Then  $\pi_\alpha^{-1}(\pi_\alpha(X_v))$ , being manifestly irreducible, closed, and  $B_-$ -invariant must again be a Schubert variety.*

*There are two cases: if  $vr_\alpha > v$ , then  $\pi^{-1}(\pi(X_v)) = X_v$  again, whereas if  $vr_\alpha < v$ , then  $\pi^{-1}(\pi(X_v)) = X_{vr_\alpha}$ . In the latter case, the map  $\pi_\alpha : X_v \rightarrow \pi_\alpha(X_v)$  is generically 1:1.*

Define a **Schubert patch**  $X_{w|v}$  as the intersection

$$X_{w|v} := X_w \cap (vN_-B/B).$$

Since  $N_-B/B$  is a copy of affine space, the Schubert patch naturally sits inside it as an affine subvariety, and both carry an action of  $T$ . It will also be useful to have the notation

$$X_{w|S} := \bigcup_{s \in S} X_{w|s} \quad \text{for any subset } S \subseteq W.$$

Schubert patches have been studied before, most obviously in Kazhdan-Lusztig theory; we include some more relevant references in section 4.3.

**Proposition 5.** (1)  $X_v^\circ \subseteq vN_-B/B$  for each  $v \in W$ .

(2)  $X_{w|v}$  is nonempty iff  $v \geq w$ .

(3) The Schubert patches  $\{X_{w|v}\}_{v \geq w}$  on a Schubert variety  $X_w$  are an open cover. In particular,  $X_w$  is normal and Cohen-Macaulay iff each  $X_{w|v}$  is.

(4) The Schubert patch  $X_{v|v}$  is just the Schubert cell  $X_v^\circ$ .

(5) If  $u \not\leq v$ , then  $X_{w|v} \cap X_u = \emptyset$ .

(6) The image of the open immersion  $\gamma : \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1 \rightarrow G/B$  from proposition 3 is  $X_1^\circ \coprod X_{r_\alpha}^\circ = X_{1|r_\alpha}$ .

*Proof.* (1) Fix  $v$ , and let  $N_1, N_2$  be as in lemma 9. Then

$$N_-vB/B = N_1N_2vB/B = v(v^{-1}N_1v)(v^{-1}N_2v)B/B.$$

Now since

$$v^{-1}N_2v = v^{-1}N_-v \cap N \subseteq N \subseteq B, \quad v^{-1}N_1v = v^{-1}N_-v \cap N_- \subseteq N_-,$$

we find

$$N_-vB/B = v(v^{-1}N_1v)B/B \subseteq vN_-B/B.$$

(2) The intersection  $X_w \cap vN_-B/B$  is a closed  $T$ -invariant subset of  $vN_-B/B$ . By lemma 9, it is only nonempty if it contains the basepoint  $vB$ , i.e. if  $vB \in X_w$ , or equivalently  $v \geq w$ .

(3) By part (1),  $\bigcup_{v \in W} vN_-B/B \supseteq \bigcup_{v \in W} N_-v/B = G/B$ . Intersecting with  $X_w$ , we get  $\{X_{w|v}\}$  as an open cover on  $X_w$ . Since normality and Cohen-Macaulayness are local conditions, it is enough to check them on an open cover.

(4) Use  $X_w = \coprod_{v \geq w} X_v^\circ$  to write

$$X_w|_w = X_w \cap wN_-B/B = (X_w^\circ \cap wN_-B/B) \cup \coprod_{v > w} (X_v^\circ \cap wN_-B/B).$$

Since  $wN_-B/B \supseteq N_-wB/B = X_w^\circ$ , the first term is the desired one, and it remains to show that  $X_v^\circ \cap wN_-B/B = \emptyset$  for  $v > w$ .

The set  $w^{-1}X_v \cap N_-B/B$  is a  $T$ -invariant closed subset of the big cell, so by lemma 9 if it is nonempty it must contain the basepoint, i.e.  $wB/B \in X_v$ . But this is only true for  $w \geq v$ , contradicted by  $v > w$ .

(5) Since  $X_w|_v = X_w \cap X_1|_v$ , it is enough to prove this for  $w = 1$ , using (2).

(6) We already computed the image to be  $\pi_\alpha^{-1}(N_-P_\alpha/P_\alpha)$  so it is plainly  $N_-$ -invariant. Hence it is determined by which  $wB$  it contains,  $w \in W$ , and in this case the only such  $w$  are  $\{1, r_\alpha\}$ . So the  $N_-$ -orbit decomposition of the image is

$$\begin{aligned} X_1^\circ \coprod X_{r_\alpha}^\circ &= X_1|_1 \coprod X_{r_\alpha}^\circ \\ &\subseteq X_1|_1 \cup r_\alpha X_1|_1 = X_1|_{1, r_\alpha} \end{aligned}$$

which gives us one of the desired inclusions.

For the opposite inclusion, we need to show  $\pi_\alpha^{-1}(N_-P_\alpha/P_\alpha) \supseteq r_\alpha X_1|_1$ . Note that

$$N_-P_\alpha = B_-P_\alpha = B_-(SL_2)_\alpha P_\alpha = P_{-\alpha}P_\alpha = (SL_2)_\alpha B_-P_\alpha$$

where  $(SL_2)_\alpha \subseteq P_\alpha$  denotes the root  $SL_2$  subgroup. Hence  $N_-P_\alpha/P_\alpha$  is  $r_\alpha$ -invariant, and so is its  $\pi_\alpha$  preimage. Therefore that preimage contains  $r_\alpha X_1|_1$ .

□

We will prove that these affine patches are normal and Cohen-Macaulay by induction on  $v$  with respect to the Bruhat order. Their study will require the following technical lemma, whose proof was sketched for us by Shrawan Kumar.

**Lemma 10.** *Let  $r_\alpha$  be a simple reflection, and  $w, v \in W$  such that  $vr_\alpha < v$ . Then*

$$(X_w \cap v \cdot X_{r_\alpha}^\circ) \times \mathbb{A}_{v, \alpha}^1 \cong X_w|_{vr_\alpha}.$$

*Proof.* Let  $N_{v, \alpha} := v(N \cap r_\alpha N_- r_\alpha)v^{-1}$  denote the  $T$ -invariant one-parameter subgroup of  $N_-$  with  $T$ -weight  $v \cdot \alpha$ . Similarly, let  $N_{-\alpha} = r_\alpha N r_\alpha \cap N_-$ . We will prove that the multiplication map

$$N_{v, \alpha} \times (X_w \cap v \cdot X_{r_\alpha}^\circ) \rightarrow X_w|_{vr_\alpha}$$

or equivalently

$$N_{v, \alpha} \times (X_w \cap vN_-r_\alpha B/B) \rightarrow X_w \cap vr_\alpha N_-B/B$$

is the desired isomorphism of schemes. Acting by  $(vr_\alpha)^{-1}$ , we may instead study

$$N_{-\alpha} \times ((r_\alpha v^{-1} \cdot X_w) \cap r_\alpha N_-r_\alpha B/B) \rightarrow (r_\alpha v^{-1} \cdot X_w) \cap N_-B/B.$$

First we confirm that this map takes values in the space claimed. Since  $N_-r_\alpha B/B \subseteq r_\alpha N_-B/B$  by proposition 5, we see  $r_\alpha N_-r_\alpha B/B \subseteq N_-B/B$  and

$$(r_\alpha v^{-1} \cdot X_w) \cap r_\alpha N_-r_\alpha B/B \subseteq (r_\alpha v^{-1} \cdot X_w) \cap N_-B/B.$$

Therefore it is enough to show that the target space is  $N_{-\alpha}$ -invariant. Obviously  $N_-B/B$  is invariant under  $N_-$ , hence under  $N_{-\alpha}$ . And  $r_\alpha v^{-1} \cdot X_w$  is  $N_{-\alpha}$ -invariant iff  $X_w$  is  $N_{v, \alpha}$ -invariant, which it is since  $v \cdot \alpha$  is a negative root. (This is where we use  $vr_\alpha < v$ .)

In the  $w = 1$  case, this map

$$N_{-\alpha} \times r_{\alpha} N_{-r_{\alpha}} B / B \rightarrow N_{-} B / B \cong N_{-}$$

is the  $v = r_{\alpha}$  case of the (latter) map in lemma 9, hence an isomorphism. Since this map takes each intersection with  $r_{\alpha}v^{-1} \cdot X_w$  into itself, each restriction

$$N_{-\alpha} \times (r_{\alpha}v^{-1} \cdot X_w) \cap r_{\alpha} N_{-r_{\alpha}} B / B \rightarrow (r_{\alpha}v^{-1} \cdot X_w) \cap N_{-} B / B$$

of this map is also an isomorphism.  $\square$

There are several possibilities for the relative positions of  $v, vr_{\alpha}, w, wr_{\alpha}$ ; only one case will need the full strength of the Geometric Vertex Decomposition Lemma. We start with the cases that don't.

**Proposition 6.** *Let  $w, v \in W$ , and assume  $v \geq w$ . Fix a simple root  $\alpha$  and the submersion  $\pi_{\alpha} : G/B \rightarrow G/P_{\alpha}$ . In the following,  $\mathbb{A}_{\beta}^1$  denotes the 1-dimensional representation of  $T$  with weight  $\beta$  and  $\mathbb{P}_{\beta}^1$  its projective completion, and all isomorphisms claimed are  $T$ -equivariant.*

1. Assume  $w < wr_{\alpha}$ . Then  $vr_{\alpha} \geq w$ , and

$$\begin{aligned} X_w|_v &\cong \pi_{\alpha}(X_w|_{v, vr_{\alpha}}) \times \mathbb{A}_{-v, \alpha}^1 \\ X_w|_{vr_{\alpha}} &\cong \pi_{\alpha}(X_w|_{v, vr_{\alpha}}) \times \mathbb{A}_{v, \alpha}^1 \end{aligned}$$

2. If  $vr_{\alpha} \not\leq w$ , then  $w > wr_{\alpha}$  and  $v > vr_{\alpha}$ . If in addition  $X_{wr_{\alpha}}|_{v, vr_{\alpha}}$  is normal (as indeed it is), then

$$X_w|_v \times \mathbb{P}_{-v, \alpha}^1 \cong X_{wr_{\alpha}}|_{v, vr_{\alpha}}.$$

3. If  $wr_{\alpha} > w$  for all simple roots  $\alpha$ , then  $w = 1$  and  $X_w|_v$  is smooth for all  $v$ .

*Proof.* For any  $u \in W$  we have  $uP_{\alpha} = ur_{\alpha}P_{\alpha}$ , so the points  $uB, ur_{\alpha}B$  lie in the same  $\mathbb{P}^1$  fiber of  $\pi_{\alpha}$ . If  $w < wr_{\alpha}$ , then  $\pi_{\alpha} : X_w \rightarrow \pi_{\alpha}(X_w)$  is a  $\mathbb{P}^1$ -bundle, so  $u \geq w \iff ur_{\alpha} \geq w$ .

1. Since we're assuming  $w < wr_{\alpha}$  and  $w \leq v$ , by the above we have  $w \leq vr_{\alpha}$  too.

Since  $w < wr_{\alpha}$ , by the BGG/Demazure proposition 4 we know  $X_w$  is a union of fibers of  $\pi_{\alpha}$ . In particular, restricting the bundle  $\pi_{\alpha}$  to the base  $v \cdot N_{-} P_{\alpha} / P_{\alpha}$  we see

$$X_w|_{v, vr_{\alpha}} \cong \pi_{\alpha}(X_w|_{v, vr_{\alpha}}) \times \mathbb{P}_{-v, \alpha}^1 \quad \text{using } \gamma \text{ from proposition 3.}$$

Twist proposition 5 part (6) by  $v, vr_{\alpha}$ :

$$X_1|_{v, vr_{\alpha}} = X_1|_v \coprod v \cdot X_{r_{\alpha}}^{\circ}, \quad X_1|_{v, vr_{\alpha}} = X_1|_{vr_{\alpha}} \coprod vr_{\alpha} \cdot X_{r_{\alpha}}^{\circ}$$

Intersect with  $X_w$  and rewrite:

$$X_w|_v = X_w|_{v, vr_{\alpha}} \setminus (v \cdot X_{r_{\alpha}}^{\circ}), \quad X_w|_{vr_{\alpha}} = X_w|_{v, vr_{\alpha}} \setminus (vr_{\alpha} \cdot X_{r_{\alpha}}^{\circ})$$

By proposition 3,  $vX_{r_{\alpha}}^{\circ}$  and  $vr_{\alpha}X_{r_{\alpha}}^{\circ}$  correspond under  $\gamma$  to the  $\infty$  and  $0$  sections. So the isomorphism above restricts to

$$X_w|_v \cong \pi_{\alpha}(X_w|_{v, vr_{\alpha}}) \times (\mathbb{P}_{-v, \alpha}^1 \setminus \{\infty\}), \quad X_w|_{vr_{\alpha}} \cong \pi_{\alpha}(X_w|_{v, vr_{\alpha}}) \times (\mathbb{P}_{-v, \alpha}^1 \setminus \{0\})$$

as desired.

2. Since  $v \in X_w$ , if  $w < wr_{\alpha}$  then  $vr_{\alpha} \in X_w$ , contradiction.

If  $v < vr_{\alpha}$ , then  $w \leq v$  implies  $w < vr_{\alpha}$ , again a contradiction.

Consider the map

$$\pi : X_w|_v \rightarrow \pi_{\alpha}(X_{wr_{\alpha}}|_{v, vr_{\alpha}})$$

restricted from  $\pi_\alpha$ . By the BGG/Demazure proposition 4, it is onto and generically 1:1, which is where we use the assumption  $vr_\alpha < w$ . Since  $X_{wr_\alpha}|_{v, vr_\alpha}$  is assumed normal, and is a (trivial)  $\mathbb{P}^1$ -bundle over the target, the target is also normal. Consider the open subset

$$U = \{u \in \pi_\alpha(X_{wr_\alpha}|_{v, vr_\alpha}) : \pi^{-1}(u) \text{ is finite}\},$$

which is also normal. Then the map  $\pi : \pi^{-1}(U) \rightarrow U$  is proper (being the restriction of the proper map  $X_w \rightarrow \pi_\alpha(X_w)$ ), finite by construction, and generically 1:1; since its target  $U$  is normal (being open in the normal variety  $\pi_\alpha(X_{wr_\alpha}|_{v, vr_\alpha})$ ) we see  $\pi : \pi^{-1}(U) \rightarrow U$  is an isomorphism.

What this shows is that the fibers of  $\pi$  that are not single (reduced) points are entire  $\mathbb{P}^1$ 's (the fibers of  $\pi_\alpha : G/B \rightarrow G/P_\alpha$ ). We now wish to show that no such fibers occur, i.e.  $\pi$  is an isomorphism.

Let  $\pi_\alpha|_{X_w} : X_w \rightarrow G/P_\alpha$  be the restriction of  $\pi_\alpha$ . Since it is  $N_-$ -equivariant, over each  $N_-$ -orbit in  $G/P_\alpha$  the fiber is constant. By the BGG/Demazure proposition 4, the  $\pi_\alpha$ -preimages of those orbits are  $X_u^\circ \coprod X_{ur_\alpha}^\circ$  for  $u < ur_\alpha$ .

Intersecting with  $X_w|_v$  to get the  $\pi$ -preimages, and using proposition 5 part (5), the  $\pi$ -preimage of  $\pi_\alpha(X_u^\circ)$  is empty unless  $u \leq v$ .

Let  $Q = \{gB \in G/B : X_w \supseteq \pi_\alpha^{-1}(gP_\alpha)\}$ , the  $N_-$ -invariant closed set of big  $\pi_\alpha|_{X_w}$  fibers. Being  $N_-$ -invariant, it has a Bruhat decomposition,

$$Q = \coprod_{u \in W : uB \in Q} X_u^\circ;$$

being closed, its  $N_-$ -orbit set  $\{u \in W : uB \in Q\}$  is closed under going up the Bruhat order.

Now we use the assumption  $vr_\alpha \not\geq w$ , to see  $v \notin Q$ . Hence  $u \in Q \implies u \not\leq v$ .

Hence  $\pi : X_w|_v \rightarrow \pi_\alpha(X_{wr_\alpha}|_{v, vr_\alpha})$  has no  $\mathbb{P}^1$ -fibers, so is a finite degree 1 proper map, hence an isomorphism.

Now we use the fact from proposition 3 that  $\pi_\alpha$  is a trivial  $\mathbb{P}^1$ -bundle over  $\pi_\alpha(X_{wr_\alpha}|_{v, vr_\alpha})$  to derive the desired isomorphism.  $\square$

**Theorem 3.** *Each Schubert patch  $X_w|_v$  is normal and Cohen-Macaulay.*

*Proof.* We may assume  $w \leq v$ , for otherwise  $X_w|_v$  is empty by proposition 5.

The proof is induction on  $v$ . If  $v = 1$ , then  $w = 1$ , and  $X_w|_v = N_-B/B$  is smooth. Otherwise we fix a simple root  $\alpha$  with  $vr_\alpha < v$  (we know one exists by the last claim in proposition 6, though it is phrased there for  $w$ ).

By proposition 6,

$$\begin{array}{lll} wr_\alpha > w & \implies & X_w|_v \cong X_{w|vr_\alpha} & \text{albeit not } T\text{-equivariantly} \\ wr_\alpha < w, \text{ but } vr_\alpha \not\leq w, & \implies & X_w|_v \times \mathbb{A}_{-v, \alpha}^1 \cong X_{wr_\alpha}|_v \cong X_{wr_\alpha}|_{vr_\alpha} \end{array}$$

and since  $vr_\alpha < v$ , by induction we know each right-hand side is normal and Cohen-Macaulay. (Indeed, we need that, to be able to invoke case (2) of proposition 6.)

We are now in the case  $v > vr_\alpha \geq w > wr_\alpha$ . (In fact  $vr_\alpha > w$  automatically, as otherwise we would have  $v > vr_\alpha > v$ .) It is here that we will finally apply the Geometric Vertex Decomposition Lemma.

Consider the open embedding  $\gamma : \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1 \rightarrow G/B$  from proposition 3, and twist it by  $v$ . By proposition 5, the image of  $v \cdot \gamma$  is  $X_1|_{v, vr_\alpha}$ . Let  $\bar{X} \subseteq \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1$  be the preimage of  $X_w$  under this map. In particular,

$$\bar{X} \cong X_w \cap X_1|_{v, vr_\alpha} = X_w|_{v, vr_\alpha}$$

so it is automatically reduced and irreducible. Our goal is to show that  $\bar{X}$  is normal and Cohen-Macaulay.

**Checking the conditions of the Geometric Vertex Decomposition Lemma.** To do so, we need first compute  $\Pi$  and  $\Lambda$ . Consider the two commuting squares

$$\begin{array}{ccccc} \text{Rad}(P_{-\alpha}) \times \mathbb{P}^1 & \xrightarrow{v \cdot \gamma} & G/B & \leftarrow & X_w \\ \downarrow & & \downarrow \pi_\alpha & & \downarrow \\ \text{Rad}(P_{-\alpha}) & \xrightarrow{v} & G/P_\alpha & \leftarrow & \pi_\alpha(X_w) = \pi_\alpha(X_{wr_\alpha}) \end{array}$$

where each  $\rightarrow$  is an open embedding and each  $\leftarrow$  is a closed embedding. Since each vertical map is (proper and) surjective, so too is the map from the pullback of the top row,  $\bar{X}$ , to the pullback of the bottom row. Hence the pullback of the bottom row is the  $\Pi \subseteq \text{Rad}(P_{-\alpha})$  we seek.

As in proposition 5, the image of  $\text{Rad}(P_{-\alpha}) \rightarrow G/P_\alpha$  is  $vN_{-P_\alpha}/P_\alpha$ , the ( $v$  twist of the) big cell. Hence the pullback  $\Pi$  of the bottom row is  $vN_{-P_\alpha}/P_\alpha$  intersected with  $\pi_\alpha(X_w) = \pi_\alpha(X_{wr_\alpha})$ . By part (1) of proposition 6 applied to  $wr_\alpha$  (not  $w$ ),

$$X_{wr_\alpha}|_{vr_\alpha} \cong \pi_\alpha(X_{wr_\alpha}|_{v, vr_\alpha}) \times \mathbb{A}_{v, \alpha}^1 = \Pi \times \mathbb{A}_{v, \alpha}^1.$$

By induction, the left-hand side is normal and Cohen-Macaulay, so  $\Pi$  is too.

The last condition left to check on  $\Pi$  is that  $\bar{X} \rightarrow \Pi$  is generically 1:1. By the assumption  $w > wr_\alpha$ , the surjection  $X_w \rightarrow \pi_\alpha(X_w)$  is generically 1:1, and the map  $\bar{X} \rightarrow \Pi$  is just a restriction of that to an open subset.

Even before we compute  $\Lambda$  exactly, we point out that  $(1, 0) \xrightarrow{v \cdot \gamma} vB \in X_w|_{v, vr_\alpha}$  by the assumption  $v \geq w$ , so  $\bar{X} \not\subseteq \text{Rad}(P_{-\alpha}) \times \{\infty\}$ .

Now we compute  $\Lambda \times \{\infty\} = \bar{X} \cap (\text{Rad}(P_{-\alpha}) \times \{\infty\})$ . Under the open embedding  $v \cdot \gamma$ ,

$$\Lambda \times \{\infty\} \cong v \cdot \gamma(\Lambda \times \{\infty\}) = X_w \cap v \cdot \gamma(\text{Rad}(P_{-\alpha}) \times \{\infty\}).$$

Here

$$\gamma(\text{Rad}(P_{-\alpha}) \times \{\infty\}) = \text{Rad}(P_{-\alpha})r_\alpha B/B = N_{-r_\alpha}B/B = X_{r_\alpha}^\circ$$

so

$$\Lambda \times \{\infty\} \cong v \cdot \gamma(\Lambda \times \{\infty\}) = X_w \cap v \cdot X_{r_\alpha}^\circ.$$

Now we use lemma 10 to relate this space  $X_w \cap v \cdot X_{r_\alpha}^\circ$  to  $X_w|_{vr_\alpha}$ , which is reduced (and irreducible) since it is an open set in  $X_w$ , and is normal and Cohen-Macaulay by induction.

With these conditions on  $\Lambda$  and the ones already checked on  $\Pi$ , we may apply the Geometric Vertex Decomposition Lemma and lemma 2, and see that  $\bar{X}$  is normal and Cohen-Macaulay.  $\square$

A stronger statement is known: for any nef line bundle  $\mathcal{L}$  on  $G/B$ , the affine variety  $\text{Spec} \bigoplus_{n \in \mathbb{N}} H^0(X_w; \mathcal{L}^{\otimes n})$  is normal and Cohen-Macaulay. (While it would seem that this property of  $(X_w, \mathcal{L})$  should be called “affinely normal” and “affinely Cohen-Macaulay”,

the regrettably standard terminology is “projectively normal” and “arithmetically Cohen-Macaulay”.) We did not see how to derive these stronger statements with the techniques of this paper.

We extract the following result from the above proof, filling in the case that was left open in proposition 6.

**Proposition 7.** *Assume  $v > vr_\alpha > w > wr_\alpha$ . Then there is a  $T$ -equivariant flat, locally free, degeneration of  $X_w|_v$  to the reduced scheme*

$$(\Pi \times \{0\}) \cup_{\Lambda \times \{\vec{0}\}} (\Lambda \times \mathbb{A}_{v \cdot \alpha}^1)$$

where

$$\Pi \times \mathbb{A}_{v \cdot \alpha}^1 \cong X_{wr_\alpha}|_{vr_\alpha} \quad \text{and} \quad \Lambda \times \mathbb{A}_{-v \cdot \alpha}^1 \cong X_w|_{vr_\alpha}, \quad T\text{-equivariantly.}$$

To see that the family constructed in theorem 2 is not just flat but locally free, we note that the  $T$ -action on the fibers contains a one-parameter subgroup  $v \cdot \check{\rho}$  acting with only positive weights. Taking various GIT quotients, one may reduce to the projective case, where flat families are automatically locally free.

**4.3. Connections to subword complexes and the Billey-Willems formula.** The proof of theorem 3 used a simple root  $\alpha$  with  $vr_\alpha < v$ . To unroll the induction, then, one needs a **reduced word** for  $v$ , which is a minimal sequence  $Q = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$  of simple roots such that  $v = \prod_{i=1}^k r_{\alpha_i}$ . (Careful: the first root used in the proof is then  $\alpha_k$ , not  $\alpha_1$ .)

In [KM04], we associated a simplicial complex  $\Delta(Q, w)$  to a (not necessarily reduced) word  $Q$  and a Weyl group element  $w$ , called the **subword complex**. The vertices are the elements of  $Q$ , and a subset of  $Q$  is a facet (maximal face) if its complement is a reduced word for  $w$ . In particular all the facets have the same dimension.

**Theorem 4.** *Let  $w \leq v$  in the Bruhat order, and let  $Q$  be a reduced expression for  $v$ .*

- (1) [B99] *The restriction of the equivariant cohomology class  $[X_w] \in H_T^*(G/B)$  to the point  $v$  can be computed as a sum over the facets of  $\Delta(Q, w)$ .*
- (2) [W06] *The restriction of the equivariant K-class  $[X_w] \in K_T^*(G/B)$  to the point  $v$  can be computed as an alternating sum over the interior faces of  $\Delta(Q, w)$ .*
- (3)  *$X_w|_v$  has a  $T$ -equivariant flat locally free degeneration to the Stanley-Reisner scheme of (an irrelevant multicone on)  $\Delta(Q, w)$ .*

All the geometric groundwork has been laid in propositions 6 and 7. The details of the bookkeeping will appear elsewhere [K], but we include a sketch here.

*Proof sketch.* The complex  $\Delta(Q, w)$  has a “vertex decomposition” into two subcomplexes, depending on whether one’s subword uses the last letter in  $Q$  or not. Things are simple when the last letter is required or forbidden, and more interesting when it is optional.

The simple cases exactly match proposition 6, and the interesting case matches the geometric vertex decomposition in proposition 7. This, and induction, prove the third claim. Since the K-class and cohomology class are invariant in locally free  $T$ -equivariant families, they can be computed from the subword complex.

In [KM04] we prove that the subword complex is homeomorphic to a ball, so its K-class can be computed as an alternating sum over the interior faces. We characterize

those faces in a way that exactly matches the terms in Willems' formula. Then either the degeneration, or Willems' formula, imply Billey's formula.  $\square$

This degeneration to the Stanley-Reisner scheme of a shellable simplicial ball generalizes ones from [KR03, GR06, KL04] which applied to Schubert patches in various types of Grassmannians. Very loosely speaking, in either situation one needs a certain multiplicity-freeness to ensure even that the degeneration is generically reduced. In the Grassmannian cases the Plücker embedding is into a minuscule representation, and the minusculeness provides the multiplicity-freeness. Here the multiplicity-freeness comes from lemma 10 and from the fact that the projection  $X_w \rightarrow \pi_\alpha(X_w)$  is generically 1:1 for  $w > wr_\alpha$ .

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